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(REVIEW ARTICLE)

Characterization of a wiener process taking values in a Hilbert space

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Abstract

This paper characterize wiener process by taking values in a Hilbert space.

A standard wiener process is stochastic process $\{W_t\}_{t\geq 0+}$ indesed by nonnegative real numbers t with the following properties:

W_0=0

With probability 1, the function t →W_t is continuous in t.

The process $\{W_t\}_{t\geq 0}$ has stationary, independent increments.

The increments W_{t+s}- W_s has the NORMAL (0,t) distribution

Keywords: Wiener process; Hilbert space; Characteristic function; Orthonormal system.

1 Introduction

Characterization theorems for wieners process by taking values in a Hilbert space have been discussed.

Renyi [1] discussed the characterization of a Wiener process taking values in a Hilbert space a follows:

Let Δ be the interval [0,1] and B denote the σ - algebra of Borel subsets of[0,1].

For each Δ ∈ B , let Φ (Δ) be a random element taking values in a real separable Hilbert space H. Suppose (Δ) satisfies the following properties.

(i) If (Φ) and (Δ′) are disjoint Borel subsets of [0,1], then Φ (Δ) and Φ (Δ′)are independent.

$$\Phi(\Delta \cup \Delta') = \Phi(\Delta) + \Phi(\Delta')$$

(ii) Φ (Δ) has stationary increments (ie) Φ (Δ) and Φ (Δ′)are identically distributed if Φ and Δ′ have the same Lebsegue measure.

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(iii) If μ_1 denotes the probability measure of Φ $[0,t]$ then μ_1 converges weakly to the distribution degenerate at the origin as $t \to 0$.

For any two $\bar{x}, \bar{y}$ in R^k

$$\langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^{k} x_j y_j$$

$$\hat{\mu}(\bar{t}) = \int e^{i\langle \bar{t}, \bar{x} \rangle} \cdot d\mu(\bar{x}), \quad \bar{t} \in R^k$$

The complex valued function μ^ on R^k is called the Fourier transform or characteristic function of the probability measure. If $\bar{f}$ is an R^k valued random variable on a probability space (Ω, s, p) and $\mu = p\bar{f}^{-1}$ is the distribution of $\bar{f}$, its characteristic function μ^is given by

$$\hat{\mu}(\bar{t}) = \int e^{i\langle \bar{t}, \bar{x} \rangle} \cdot d\mu(\bar{x})$$

$$= \int e^{i\langle \bar{t}, \bar{f} \rangle} \cdot dp$$

$$= E[e^{i\langle \bar{t}, \bar{f} \rangle}]$$

μ^is the characteristics function of the random variable $\bar{f}$

2 Proposition 1

The multivariate normal distribution in R^k with mean vector $\bar{m}$ and co – variance matrix Σ has characteristic function $e^{i\langle \bar{t}, \bar{t} \rangle} - 1/2\, t^{-1} \sum \bar{j}$.

2.1 Definition 1

2.1.1 Orthonormal System

A sequence ξ_n ($n = 1,2, \ldots \ldots .$) of random variable on a probability space $S = (\Omega, A, P)$ is called on orthonormal system, if ξ_n belongs to the Hilbert space $L_2(S)$.

(ie).,

(ξ^2_n) exists and has

$(\xi^2_n) = 1, n = 1,2, \ldots \ldots \ldots$

$[\xi_n \ \xi_m] = 0, n \neq m$

2.2 Definition 2

2.2.1 Hilbert Space

A Banach space is called a Hilbert space, if the function (x, y) inner product of x and y has the following properties.

1. $(x, y) = (y, x)$
2. $(x, x) = \|x\|^2$
3. For fixed $y, (x) = (x, y)$ is a linear functional. (i.e) $[A(ax + by)] =$

   $aA(x) + b(A(y))$

2.3 Definition 3

Complete Orthonormal System

The orthonormal system {} is complete if $\eta \in L_2(s), E(\eta\xi_n) = 0$,

$n = 1,2, \ldots \ldots$ it follows that $\eta = 0$ almost surely.

2.4 Definition 4

2.4.1 Fourier Co – efficient

If {} is an orthonormal system on S and η is an arbitrary random variable

$\eta \in L_2(s)$, the sequence $C_n = E[\eta\xi_n]$ is called the sequence of fourier co – efficient of η and the series $\sum C_n\xi_n$ is fourier series of η with respect to $\{\xi_n\}$.

$$\sum_{n=1} C_n^2 = E(\eta^2)$$

is Parseval's relation.

2.5 Definition 5

2.5.1 Rademacher Function

Let S be Lebesgue probability space and consider the Rademacher function.

$$R_n(x) = (sin\, 2^n\pi x)\ (0 \leq x \leq 1, n = 1,2, \ldots\ldots)$$

2.6 Definition 6

2.6.1 Walsh Functions

Let us now define the functions $W_n(x), 0 \leq x \leq 1, n = 0,1,2, \ldots\ldots$ as follows.

Let $W_0(x), 0 \leq x \leq 1$.

Further if the representation of $n \geq 1$ in the binary system is $n = 2^{k1} + 2^{k2} + \ldots\ldots\ldots + 2^{kr}$ are integer

Put $W_n(x) = R_{k_1+1}(x)\, R_{k_2+1}(x)$.. $R_{k_r+1}(x)$.

The function $W_n(x)$, [n = 0,1,2,..] are called Walsh functions.

To prove that $\{w_n(x)\}$ form an orthonormal system on the Lebesgue probability space. If $\xi_1, \xi_2, \ldots\ldots.., \xi_n$ are independent random variables with finite expectation then $\xi_1, \xi_2, \ldots\ldots.., \xi_n$ also has finite expectation and

$$(\xi 1, \xi 2, \; , \xi n) = \mathrm{G}_{k=1}^{n} E(\xi k).$$

bability space.

3 Theorem 1

If the random variables ξn are independent (n = 1,2, ...), E(ξ_n^2) =1and $E(\xi n) = 0$ for n = 1,2, ….. then all the products $\xi k_1, \xi k_2, \; , \xi k_r$ ($1 \le k1 < k2 < \ldots\ldots < kr, r = 1,2,$) belong to $L2(s)$ and they form together with the constant 1 and orthonormal system.

Proof

We have $\boldsymbol{\xi_{k1}, \xi_{k2}, \ldots \xi_{kr} = \prod_{j=1}^{r} E(\xi_{k^2}) = 1}$

If we take any two non identical product $\xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_r}$.

$(k_1 < k_2 < \ldots\ldots < k_r)$ and $\xi l_1, \xi l_2, \ldots\ldots.., \xi l_s$ $(l_1 < l_2 <$...$<$).

$(\xi_n) = 0, n \ge 1$ and their product has expectation zero.

Then we have $(l^2) = 1$.

$$(1 \cdot \xi_{k_1}, \xi_{k_2}, \ldots\ldots\ldots, \xi_{k_r}) = 0$$

Next to prove that this system is complete.

Let x be a real number $0 \le x < 1$ which is not a binary rational number.

Let the binary expansion of x be $x = \sum_{k=1}^{\infty} \frac{\xi_k(x)}{2^k}$ ($\xi_k(x)$=0 or 1)

$$\text{Then } \xi_k(x) = \frac{1 - R_k(x)}{2}$$

(x) = 1 or −1 when $\xi(x)$ = 0 or 1.

Let i(x) denotes the indicator function of the interval $(\frac{m}{2n}, \frac{m+1}{2n})$ where m and n are non negative integers and $0 \le m < 2^n$.

Let the binary expantion of m / 2^n be

$$\frac{m}{2_n} = \sum_{k=1}^{n} \frac{\delta_k}{2_k} \quad (\delta_k = 0 \; or \; 1, k = 1,2,$$

Then $i_{n,}(x)$ can be written in the form

$$i_{n,m}(x) = \sum_{l=0}^{2^n-1} a_{n,m,l} W_l(x)$$

It follows that if $f = (x) \in L_2(s)$ is such a function that

$(fw_n) = 0, n = 0,1, \dots \dots$

Then we have

$m{+}1/\,2n$

$\int f(x)dx = 0, 0 \leq m < 2^n, \quad n = 1,2, \dots \dots$

$m/2\mathrm{n}$

$m{+}1/\,2n$

$\int f(x)dx = 0, 0 \leq m < 2^n, \quad n = 1,2, \dots \dots$

0

$$Thus\, putting\, (x) = \int_0^t (t) = 0.$$

We get $(r) = 0$ for every binary rational number r in (0,1). The function (x) being the indefinite integral of an integrable function is continuous thus $(x) = 0$ for all X in [0,1].

Therefore $(x) = 0$ for almost all x.

Therefore the system of Walsh functions is complete.

The series defining (t) is almost surely convergent, because denoting by $e_t(x) = \binom{0<x<1}{0<t<1}$ the indicator of the interval $(0,t)$ and taking into account that

$$\int_0^t W_n(x)dx = \int_0^l e_t(x) w_n(x)dx$$

and Fourier Walsh co – efficient of function $e_t(x)$. Since $\{(x)\}$ is a complete orthonormal system. We get from Parseval's relation,

$$\sum_{n=0}^{\infty} \left(\int_0^t W_n(x)dx\right)^2 = t$$

$Therefore\ (\eta 2(t)) = E\Sigma_n \xi 2(w)(\int W_n(x)dx) = t\ [sinPe\ E(\xi_n 2) = 1$

The Parseval's relation $\Longrightarrow 0 < s < t < 1$

$$[\eta(s)\eta(t)] = E\Sigma_n \xi^2_n \left(\int_0^t W_n(x)\right)\left(\int_0^s W_n(x)dx\right)$$

$$= \Sigma \left(\int_0^t W_n(x)dx\right)\left(\int_0^s W_n(x)dx\right) = \Sigma \left(\int_0^t W_n(x)dx\right)$$

$$= \int_0^1 e_s(x)e_t(x)dx = s$$

$$[(t_1) - (s_1)][(t_2) - (s_2)] = t_1 - s_1 - t_1 + s_1$$

$$= 0$$

$$[((t) - (s))^2] = t + s - 2s = t - s \text{ if } s < t$$

Similarly we get that the joint distribution of $((t\mathrm{W}) - \eta(s\mathrm{W}))\ 1 \le \mathrm{W} \le k$ is a k dimensional normal distribution. As the components of a k dimensional normally distributed vector are independent if and only if they are uncorrelated.

It follows that $(t\mathrm{W}) - \eta(s\mathrm{W})$ $(\mathrm{W} = 1,2,\dots\dots, k)$ are independent. The almost sure continuity of (t) as a function of t can be proved as follows:-

If $2^s \le n < 2^{s+1}$.

$W_n(x) = \underline{R_{s+1}}(x)\ _{-2}S(x)$ where $W_{n-2}S(x)$ is a product of the Rademacher functions $R_k(x), k \le s$ and thus is constant on every interval of the form $(\frac{r}{2^s}, \frac{r+1}{2^s})$

On the other hand the indefinite integral of $R_{S+1}(x)$ over $(\frac{r}{2^s}, \frac{r+1}{2^s})$ increase linearly from increases linearly from zero to $\frac{1}{2^{s+1}}$ and the decrease linearly to zero.

If follows that

$$\sum_{n=2_s}^{2_{s+1}-1} \xi \qquad \int_0^t W_n(x)dx$$

is for fixed $\omega \in \Omega$ a continuous function of t such that on every interval $(\frac{r}{2^s}, \frac{r+1}{2^s})$ varies between

$$\pm \sum_{n=2^s}^{2^{s+1}-1} \xi_n \epsilon_{nr}$$

Now the sum $\sum_{n=2^s}^{2^{s+1}-1} \xi_n \epsilon_{nr}$is normally distributed with variance 2^s

Thus put $\delta_{st} = \sum_{n=2^s}^{2^{s+1}-1} \xi_n \epsilon_{nr}$

$P[\delta_{st} > s2^{s/2}] < e^{-s^2/2}$

Borel cantelli lemma $\sum_{s=1}^{2} e^{-s^2/2}$ is convergebt for almost all $\omega \epsilon \Omega$

$\max_{1 \leq r < 2^s} |\delta_{st}| < s2^{s/2}$ for all but a finite number of values of s.

This implies that for almost all values of ω, one has uniformly for $0 \leq t < 1$

$$\sum_{s=1}^{\infty} \left| \sum_{n=2^s}^{2^{s+1}} \xi_n(\omega) \int_0^t W_n(x)dx \right| < \sum_{s=1}^{\infty} \frac{s}{2^{s/2+1}} < \infty$$

Therefore $\eta(t)$ is for almost all ω the sum of a uniformly convergent series

of continuous function. (i.e.,) it is almost surely a continuous function of t.

Compliance with ethical standards

Disclosure of conflict of interest

I have no conflict of interest to be disclosed.

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